

Linear Relations of Siegel Poincaré Series and Non-vanishing of the Central Values of Spinor L-functions

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Abstract In this paper, we will first investigate the linear relations of a one parameter family of Siegel Poincaré series. Then we give the applications to the non-vanishing of Fourier coefficients of Siegel cusp eigenforms and the central values.

Keywords Siegel Poincaré series, Fourier coefficients of Siegel cusp forms, non-vanishing of central values, Böcherer conjecture.

Mathematics Subject Classification 11F46, 11F30, 11F67

1 Introduction

Fourier coefficients of cusp forms are fundamental objects in number theory and there are many open questions. One basic problem is to determine whether a Fourier coefficient vanishes or not. Such problems have a significant application in the theory of L-functions. A remarkable example are the $GL(2)$ cusp forms. Indeed, due to Waldspurger's theorem, Fourier coefficients of a half-integral weight cusp form will determine the central values of L-functions of the corresponding integral weight cusp form.

To study the Fourier coefficients of cusp forms, an effective tool are the Poincaré series. It is known that the space of holomorphic $SL(2, \mathbb{Z})$ cusp forms is a vector space of finite dimension. Additionally, the Poincaré series can determine Fourier coefficients of cusp forms via the Petersson inner product and hence the space can be generated by Poincaré series. Therefore, non-vanishing results and linear relations of Poincaré series will reveal the information for Fourier coefficients.

Such problems have been investigated by several authors. For the non-vanishing of Poincaré series, one can refer to [Ra], [Mo]. For the linear relations of Poincaré series, Petersson's paper [Pe] showed that, the first m Poincaré series span the (finite-dimensional) vector space of weight k full level holomorphic cusp forms, where m is the dimension of the space. Later, Petersson's result was generalized to Hecke congruence groups by Lehner [Le] under some conditions.

In this paper, we will consider the situation of Siegel cusp forms. Similar to the $GL(2)$ case, non-vanishing results and linear relations of Siegel Poincaré series are effective tools for studying the Fourier coefficients of Siegel cusp forms, especially Hecke Siegel cusp forms.

The non-vanishing of Siegel Poincaré series was studied by several papers. (See [DS], [DKS].) In our paper, we will investigate the linear relations of a one parameter family of Siegel Poincaré series.

We proceed to describe our results. Let $\Gamma_2 = Sp_4(\mathbb{Z})$. Assume that k is an even number and F a Siegel cusp form of weight k . This is a function on Siegel's half plane $\mathbb{H}_2 = \{Z = X + iY \in Mat_2(\mathbb{C}) | Z = {}^tZ, Y > 0\}$ satisfying

$$F(\gamma Z) = F((AZ + B)(CZ + D)^{-1}) = (\det J(\gamma, Z))^k F(Z) \quad \text{for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{Z})$$

where $J(\gamma, Z) = CZ + D$ and some holomorphy conditions on \mathbb{H}_2 . For a general introduction to Siegel cusp forms, see [Kl], [Pi]. Denote by $\mathcal{S}_k(\Gamma_2)$ the vector space generated by Siegel cusp forms of weight k . It is known that $\mathcal{S}_k(\Gamma_2)$ has finite dimension. By the automorphy and the holomorphy of F , it has Fourier expansion

$$F(Z) = \sum_T A(F, T) e(\text{Tr}(TZ)),$$

where the summation is over all symmetric, positive definite and half-integral matrices T . We equip the space $\mathcal{S}_k(\Gamma_2)$ with the Petersson inner product:

$$\langle F, G \rangle = \int_{\Gamma_2 \backslash \mathbb{H}_2} F(Z) \overline{G(Z)} (\det Y)^k \frac{dXdY}{(\det Y)^3}.$$

By the Hecke theory, we can find an (orthogonal) basis, denoted by \mathcal{B}_k , of $\mathcal{S}_k(\Gamma_2)$ which are eigenfunctions of Hecke operators. Then for $F \in \mathcal{B}_k$, we can define the spinor L-function, denoted by $L(s, \text{spin}, F)$. By some normalization, we can assume that the critical strip is $0 < \text{Re } s < 1$. (Indeed, this is the spinor L-function of π_F , which is the automorphic cuspidal representation associated to F . See Chapter 6 in [Pi].)

In addition, the space $\mathcal{S}_k(\Gamma_2)$ contains a subspace of (Saito-Kurokawa) lifts from holomorphic $SL(2, \mathbb{Z})$ cusp forms of weight $2k - 2$. Denote by \mathcal{K} the subspace of Saito-Kurokawa lifts. Furthermore, let f be an $SL(2, \mathbb{Z})$ Hecke eigenform of weight $2k - 2$. Then the corresponding Siegel cusp form F_f is also a Hecke eigenform. The spinor L-function of F_f and the (normalized) L-function of f are connected by the following relation:

$$L(s, \text{spin}, F_f) = \zeta(s + 1/2) \zeta(s - 1/2) L(s, f).$$

Denote by Γ_∞ the group of translations in Γ_2 , that is,

$$\Gamma_\infty = \left\{ \begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) \mid X = {}^t X \right\}.$$

In the theory of Siegel cusp forms, the Poincaré series are defined by

$$\mathbb{P}_Q(Z) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{Sp}_4(\mathbb{Z})} \det(J(\gamma, Z))^{-k} e(\mathrm{Tr}(Q\gamma Z)) = \sum_T A(\mathbb{P}_Q, T) e(\mathrm{Tr}(TZ))$$

for any Q , a symmetric, positive definite and half-integral matrix. (The explicit form of $A(\mathbb{P}_Q, T)$ will be given in Lemma 2.1) The following formula explains why Siegel Poincaré series can determine Fourier coefficients of Siegel cusp forms ([KST], (3.1.1)):

$$\langle F, \mathbb{P}_Q \rangle = 8c_k \frac{A(F, Q)}{(\det Q)^l} \quad (1.1)$$

where

$$l = k - 3/2 \quad \text{and} \quad c_k = \frac{\pi^{1/2}}{4} (4\pi)^{3-2k} \Gamma(k - 3/2) \Gamma(k - 2).$$

In this paper, we will study Siegel Poincaré series \mathbb{P}_Q with $Q = nI_2$ for n a positive integer. We will establish the following theorem:

Theorem 1.1. Let $\epsilon > 0$. Denote by \mathcal{V} the vector space spanned by \mathbb{P}_{nI_2} . Then for sufficiently large k , we can find a subspace $\mathcal{W} \subseteq \mathcal{V}$, such that

$$\dim \mathcal{W} \geq k^{2/3-\epsilon},$$

and

$$\mathcal{W} \cap \mathcal{K} = \emptyset.$$

In section 3, the construction of the subspace \mathcal{W} will be given explicitly. To prove Theorem 1.1, we need to investigate the matrix \mathcal{A} defined in 3.1. We will show that the matrix \mathcal{A} is invertible and this will give the linear independence of Siegel Poincaré series. For the second part, we need the Maass relation 3.2.

As an application, we will study the number of Hecke eigenforms F satisfying $A(F, I_2) \neq 0$ in Section 4. We can establish the following result:

Corollary 1.2. Let $\epsilon > 0$ be arbitrary. Then for sufficiently large k ,

$$\#\{F \in \mathcal{B}_k \mid F \text{ is not a Saito-Kurokawa lift and } A(F, I_2) \neq 0\} \geq k^{2/3-\epsilon}.$$

In [Bo], Böcherer made a remarkable conjecture that $A(F, I_2)$ should be related to central L-values. This can be regarded as a generalization of Waldspurger's theorem. A precise version of Böcherer's conjecture was given in [DPSS]. In our case, the statement is, for a non-Saito-Kurokawa lift F ,

$$\frac{\pi^{1/2}}{4} (4\pi)^{3-2k} \Gamma(k-3/2) \Gamma(k-2) \frac{A(F, I_2)^2}{\|F\|^2} = \frac{64\pi^6 \Gamma(2k-4)}{\Gamma(2k-1)} \frac{L(1/2, \text{spin}, F) L(1/2, \text{spin}, F \times \chi_{-4})}{L(1, \pi_F, \text{Ad})}$$

where $L(s, \pi_F, \text{Ad})$ is a degree 10 L-function. By [PSS] Theorem 5.2.1, we know that $L(1, \pi_F, \text{Ad}) \neq 0$. Later Böcherer's conjecture was proved by Furusawa and Morimoto [FM]. In our case, we have the following corollary:

Corollary 1.3. Let $\epsilon > 0$ be arbitrary. Then for sufficiently large k ,

$$\#\{F \in \mathcal{B}_k \mid F \text{ is not a Saito-Kurokawa lift and } L(1/2, \text{spin}, F) L(1/2, \text{spin}, F \times \chi_{-4}) \neq 0\} \geq k^{2/3-\epsilon}.$$

Remark 1.4. In Blomer's paper [Bl], he proved that, for sufficiently large k , there exists one Hecke eigenform F (not a Saito-Kurokawa lift) such that

$$L(1/2, \text{spin}, F) L(1/2, \text{spin}, F \times \chi_{-4}) \neq 0.$$

(Corollary 5 with $q_1 = q_2 = 1$.) In Corollary 4 of [Bl], Blomer established a much stronger quantitative lower bound for the size of the set $(k^{3-\epsilon})$ under the “no-Siegel-zero” hypothesis for the adjoint L-function.

2 Preliminaries

In this paper, we will always assume k to be an even integer and $k \geq 6$. Then we set $l = k - \frac{3}{2}$. We also set

$$c_k = \frac{\pi^{1/2}}{4} (4\pi)^{3-2k} \Gamma(k-3/2) \Gamma(k-2).$$

This paper relies heavily on the estimation of the Fourier coefficients of Siegel Poincaré series. We will use some results in Section 3 of Blomer's paper [Bl].

The explicit form of the Fourier coefficient of Siegel Poincaré series was calculated by Kitaoka [Ki]. Here we introduce some notations and quote his results. Let Q, T be positive definite, symmetric and half-integral matrices. Then for an invertible matrix $C \in \text{Mat}_2(\mathbb{C})$, we define the symplectic Kloosterman sum

$$K(Q, T; C) = \sum e(\text{Tr}(AC^{-1}Q + C^{-1}DT))$$

where the summation is over matrices $\left\{ \begin{pmatrix} A & * \\ C & D \end{pmatrix} \mid \begin{pmatrix} A & * \\ C & D \end{pmatrix} \in \text{Sp}_4(\mathbb{Z}) \right\}$ for a given value C in a system $X(C)$ of representatives for $\Gamma_\infty \backslash \text{Sp}_4(\mathbb{Z}) / \Gamma_\infty$. One can see that $|X(C)| \leq |\det C|^{3/2}$

and hence the sum is trivially bounded by $|\det C|^{3/2}$. For the details, one can see the argument below (3.1) in Blomer's paper [Bl].

Let P be a real and positive definite 2×2 matrix with eigenvalues s_1^2 and s_2^2 , then we define

$$\mathcal{J}_l(P) = \int_0^{\pi/2} J_l(4\pi s_1 \sin \theta) J_l(4\pi s_2 \sin \theta) \sin \theta d\theta.$$

For $P = \begin{pmatrix} p_1 & p_2/2 \\ p_2/2 & p_4 \end{pmatrix}$ and $S = \begin{pmatrix} s_1 & s_2/2 \\ s_2/2 & s_4 \end{pmatrix}$ being positive definite, symmetric and half-integral matrices and $c \in \mathbb{N}$, we define another Kloosterman sum

$$H^\pm(P, S; c) = \delta_{s_4=p_4} \sum_{d_1 \pmod{c}}^* \sum_{d_2 \pmod{c}} e \left(\frac{\overline{d_1} s_4 d_2^2 \mp \overline{d_1} p_2 d_2 + s_2 d_2 + \overline{d_1} p_1 + d_1 s_1}{c} \mp \frac{p_2 s_2}{2c s_4} \right).$$

Here we will only use the trivial bound $|H^\pm(P, S; c)| \leq c^2$.

The following lemma is due to Kitaoka [Ki]:

Lemma 2.1. Let $k \geq 6$ be even. Then for P, Q being positive definite, symmetric and half-integral matrices,

$$\begin{aligned} A(\mathbb{P}_Q, T) &= \delta_{Q \sim T} \# \text{Aut}(T) \\ &+ \left(\frac{\det T}{\det Q} \right)^{l/2} \sum_{\pm} \sum_{U, V} \sum_{s, c \geq 1} \frac{(-1)^k \sqrt{2}\pi}{c^{3/2} s^{1/2}} H^\pm(UQ^tU, V^{-1}T^tV^{-1}, c) J_l \left(\frac{4\pi \sqrt{\det TQ}}{cs} \right) \\ &+ 8\pi^2 \left(\frac{\det T}{\det Q} \right)^{l/2} \sum_{\det C \neq 0} \frac{K(Q, T; C)}{|\det C|^{3/2}} \mathcal{J}_l(TC^{-1}Q^tC^{-1}). \end{aligned}$$

where the sum over $U, V \in \text{GL}_2(\mathbb{Z})$ in the second term on the right hand side is over matrices

$$U = \begin{pmatrix} * & * \\ u_3 & u_4 \end{pmatrix} / \{\pm I_2\} \quad V = \begin{pmatrix} v_1 & * \\ v_3 & * \end{pmatrix} \quad \begin{pmatrix} u_3 & u_4 \end{pmatrix} Q \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -v_3 & v_1 \end{pmatrix} T \begin{pmatrix} -v_3 \\ v_1 \end{pmatrix} = s.$$

$Q \sim T$ means the equivalence in the sense of quadratic forms and $\text{Aut}(T) = \{U \in \text{GL}_2(\mathbb{Z}) \mid U^t T U = T\}$.

Generally, these three terms are called rank 0, rank 1 and rank 2 terms respectively.

We also recall a basic lemma in Blomer's paper ([Bl] Section 3, Lemma 2):

Lemma 2.2. For positive definite matrices T, Q with largest eigenvalues λ_T, λ_Q , the smallest eigenvalue of $TC^{-1}Q^tC^{-1}$ is $\ll \lambda_T \lambda_Q \|C\|^{-2}$ where $\|\cdot\|$ denotes any submultiplicative matrix norm.

Additionally, we need the following uniform bounds for Bessel functions:

$$J_l(x) \ll \left(\frac{cx}{l} \right)^l, \quad (2.1)$$

where c can be chosen to be any constant larger than $\frac{e}{2}$. This is valid for $x > 0$ and $l > 1/2$. This follows from [[GR], 8.411.4].

Let $Y = k^{2/3-\epsilon/2}$ and we set \mathcal{P} to be the set of all primes in the interval $[Y, 2Y]$.

Lemma 2.3. For $p, q \in \mathcal{P}$, we have:

$$A(\mathbb{P}_{pI_2}, qI_2) = 8\delta_{p,q} + O\left(\frac{M^l}{l^{(1/3+\epsilon/2)l}}\right).$$

Here M is a constant independent from k .

Proof By Kitaoka's formula, we have

$$\begin{aligned} A(\mathbb{P}_{pI_2}, qI_2) &= \delta_{pI_2 \sim qI_2} \# \text{Aut}(qI_2) \\ &+ \left(\frac{\det(qI_2)}{\det(pI_2)}\right)^{l/2} \sum_{\pm} \sum_{U,V} \sum_{s,c \geq 1} \frac{(-1)^k \sqrt{2}\pi}{c^{3/2} s^{1/2}} H^{\pm}(pU^t U, qV^{-1t} V^{-1}, c) J_l\left(\frac{4\pi \sqrt{\det(pqI_2)}}{cs}\right) \\ &+ 8\pi^2 \left(\frac{\det(qI_2)}{\det(pI_2)}\right)^{l/2} \sum_{\det C \neq 0} \frac{K(pI_2, qI_2; C)}{|\det C|^{3/2}} \mathcal{J}_l(pqC^{-1t} C^{-1}). \end{aligned}$$

Notice that $pI_2 \sim qI_2$ if and only if $p = q$. On the other hand, $\# \text{Aut}(pI_2) = \# \text{Aut}(I_2) = 8$. This handled the rank 0 term.

Next, by the definition of the exponential sum in the rank 1 term, we know that $s = p(u_3^2 + u_4^2) = q(v_1^2 + v_3^2)$ and hence $[p, q] | s$. Here $[p, q]$ is the least common multiple. Then by the change of variable $s \mapsto [p, q]s$, the estimation (2.1) and the fact that $p, q \in [Y, 2Y]$ we know that the rank 1 term is bounded by

$$\frac{M^l}{l^{(1/3+\epsilon/2)l}}.$$

Finally, we consider the rank 2 term. We know that $K(pI_2, qI_2; C)$ is bounded by $|\det(C)|^{3/2}$. By Lemma 2.2, the smallest eigenvalue of $pqC^{-1t} C^{-1}$ is $<< \frac{pq}{\|C\|^2}$, where $\|C\| = \text{Tr}(C'C)$ is a matrix norm on 2×2 matrices. This shows that

$$\mathcal{J}(pqC^{-1t} C^{-1}) << \left(\frac{M'(pq)^{1/2}}{\|C\|l}\right)^l$$

for some constant M' independent from k . Then the rank 2 term is bounded by

$$\frac{M^l}{l^{(1/3+\epsilon/2)l}}.$$

□

Lemma 2.4. For $p, q \in \mathcal{P}$, we have:

$$A\left(\mathbb{P}_{pI_2}, \begin{pmatrix} q^2 & \\ & 1 \end{pmatrix}\right) << \frac{M^l}{l^{3\epsilon l/4}}.$$

Here M is a constant independent from k .

Proof The proof can be quite similar to Lemma 2.3. Here I would only mention two differences: for the rank 1 term, we can only do the substitution $s \mapsto ps$. For the rank 2 term, the smaller eigenvalue of $p \begin{pmatrix} q^2 & \\ & 1 \end{pmatrix} C^{-1} C^{-1}$ is bounded by

$$\frac{q^2 p}{\|C\|^2}$$

by Lemma 2.2. □

We also need the following lemma:

Lemma 2.5. For $p \in \mathcal{P}$, we have:

$$A(\mathbb{P}_{pI_2}, I_2) << \frac{M^l}{l^{(4/3-\epsilon/4)l}}.$$

Here M is a constant independent from k .

Proof The proof is similar to Lemma 2.3. □

3 Proof of Theorem 1.1

Proof Let $Y = k^{2/3-\epsilon/2}$ and we set \mathcal{P} to be the set of all primes in the interval $[Y, 2Y]$. Set $n = \#\mathcal{P}$. Then by prime number theory, $n \geq k^{2/3-\epsilon}$ when k is sufficiently large.

We need to show two things: the set $\{\mathbb{P}_{pI_2} | p \in \mathcal{P}\}$ is linearly independent. Then let \mathcal{W} be the subspace of $\mathcal{S}_k(\Gamma_2)$ spanned by $\{\mathbb{P}_{pI_2} | p \in \mathcal{P}\}$ and we need to show that $\mathcal{W} \cap \mathcal{K} = \emptyset$.

3.1 The linear independence

To show the linear independence of $\{\mathbb{P}_{pI_2} | p \in \mathcal{P}\}$, we consider the following matrix

$$\mathcal{A} = (a_{p,q})_{p,q \in \mathcal{P}} = (A(\mathbb{P}_{pI_2}, qI_2))_{p,q \in \mathcal{P}}, \quad (3.1)$$

that is, $a_{p,q} = A(\mathbb{P}_{pI_2}, qI_2)$, the qI_2 -th Fourier coefficient of \mathbb{P}_{pI_2} . We want to show that the matrix \mathcal{A} is invertible and hence Siegel Poincare series $\{\mathbb{P}_{pI_2} | p \in \mathcal{P}\}$ are linearly independent in \mathcal{V} . This can be obtained by using Lemma 2.3. Indeed, we can write

$$\mathcal{A} = 8I_n + \mathcal{B}$$

such that every entry of \mathcal{B} is uniformly bounded by $M^l / l^{(1/3+\epsilon/2)l} << \frac{1}{k^{100}}$ when k is sufficiently large. Then \mathcal{A} is invertible.

Remark 3.1. *Indeed, by a more delicate estimation, we can show, there exists a subset \mathcal{Q} of natural numbers with $\#\mathcal{Q} \gg k$ such that*

$$\{\mathbb{P}_{nI_2} | n \in \mathcal{Q}\}$$

is a linearly independent set. We will discuss this in the last section.

3.2 The Non-intersection of \mathcal{W} and \mathcal{K}

We prove this by contradiction. Suppose not, we can find $(\lambda_p)_{p \in \mathcal{P}}$ such that

$$F = \sum_{p \in \mathcal{P}} \lambda_p \mathbb{P}_{pI_2} \in \mathcal{K}$$

and not all λ_p are 0. Then without loss of generality, we can assume that $|\lambda_p| \leq 1$ for all $p \in \mathcal{P}$ and there exists a p_0 such that $|\lambda_{p_0}| = 1$.

Since $F \in \mathcal{K}$, we know that F satisfies the Maass relation:

$$A\left(F, \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix}\right) = \sum_{d|(m,n,r)} d^{k-1} A\left(F, \begin{pmatrix} \frac{mn}{d^2} & \frac{r}{2d} \\ \frac{r}{2d} & 1 \end{pmatrix}\right) \quad (3.2)$$

Then the Maass relation gives:

$$A(F, p_0 I_2) = A\left(F, \begin{pmatrix} p_0^2 & \\ & 1 \end{pmatrix}\right) + p_0^{k-1} A(F, I_2).$$

This gives:

$$\sum_{p \in \mathcal{P}} \lambda_p A(\mathbb{P}_{pI_2}, p_0 I_2) = \sum_{p \in \mathcal{P}} \lambda_p A\left(\mathbb{P}_{pI_2}, \begin{pmatrix} p_0^2 & \\ & 1 \end{pmatrix}\right) + \sum_{p \in \mathcal{P}} \lambda_p p_0^{k-1} A(\mathbb{P}_{pI_2}, I_2).$$

Therefore, we obtain

$$\begin{aligned} |\lambda_{p_0} A(\mathbb{P}_{p_0 I_2}, p_0 I_2)| &\leq \sum_{\substack{p \in \mathcal{P} \\ p \neq p_0}} |\lambda_p| |A(\mathbb{P}_{pI_2}, p_0 I_2)| \\ &\quad + \sum_{p \in \mathcal{P}} |\lambda_p| \left| A\left(\mathbb{P}_{pI_2}, \begin{pmatrix} p_0^2 & \\ & 1 \end{pmatrix}\right) \right| \\ &\quad + \sum_{p \in \mathcal{P}} |\lambda_p| p_0^{k-1} |A(\mathbb{P}_{pI_2}, I_2)|. \end{aligned}$$

Then combine Lemma 2.3, Lemma 2.4 and Lemma 2.5, and we obtain:

$$\left\{ 1 + o\left(\frac{1}{k^{100}}\right) \right\} |\lambda_{p_0}| = o\left(\frac{M^k}{k^{k\epsilon/2}}\right),$$

when k is large. This shows that $|\lambda_{p_0}| < 1$ when k is large. A contradiction to the assumption that $|\lambda_{p_0}| = 1$. \square

4 Proof of Corollary 1.2

Proof Here we need to decompose \mathcal{B}_k into four disjoint sets, denoted by \mathcal{B}_k^i and $i = 1, 2, 3, 4$. That is,

$$\mathcal{B}_k = \mathcal{B}_k^1 \cup \mathcal{B}_k^2 \cup \mathcal{B}_k^3 \cup \mathcal{B}_k^4,$$

where

$$\begin{aligned}\mathcal{B}_k^1 &= \{F \in \mathcal{B}_k | F \text{ is a Saito-Kurokawa lift and } A(F, I_2) \neq 0.\} \\ \mathcal{B}_k^2 &= \{F \in \mathcal{B}_k | F \text{ is not a Saito-Kurokawa lift and } A(F, I_2) \neq 0.\} \\ \mathcal{B}_k^3 &= \{F \in \mathcal{B}_k | F \text{ is a Saito-Kurokawa lift and } A(F, I_2) = 0.\} \\ \mathcal{B}_k^4 &= \{F \in \mathcal{B}_k | F \text{ is not a Saito-Kurokawa lift and } A(F, I_2) = 0.\}\end{aligned}$$

Denote by \mathcal{M} of $\mathcal{S}_k(\Gamma_2)$ spanned by $\mathcal{B}_k^3 \cup \mathcal{B}_k^4$. Let $F \in \mathcal{B}_k$. By Andrianov's formula,

$$\zeta(s+1/2)L(s+1/2, \chi_{-4}) \sum_m \frac{A(F, mI_2)}{m^{s+k-3/2}} = L(s, \text{spin}, F)A(F, I_2),$$

(see [An1] Theorem 2.4.1 or [An2] Theorem 4.3.16 with $l = a = 1$, and $\gamma = \eta = \chi = \text{triv}$) we see that $F \in \mathcal{B}_k^3 \cup \mathcal{B}_k^4$ implies that F is orthogonal to the space \mathcal{V} by 1.1. This will give:

$$\mathcal{V} \subseteq \mathcal{M}^\perp.$$

Here \mathcal{M}^\perp is the orthogonal complement of \mathcal{M} with respect to the Petersson inner product. Let \mathcal{K}_0 be the subspace of \mathcal{K} spanned by \mathcal{B}_k^1 . The relation above can be refined to

$$\mathcal{V} + \mathcal{K}_0 \subseteq \mathcal{M}^\perp.$$

Remark 4.1. Notice that this is only a sum of two spaces but not necessary a direct sum.

This gives:

$$\#\mathcal{B}_k^1 + \#\mathcal{B}_k^2 = \#\{F \in \mathcal{B}_k | A(F, I_2) \neq 0\} \geq \dim(\mathcal{V} + \mathcal{K}_0) \geq \dim(\mathcal{W} + \mathcal{K}_0). \quad (4.1)$$

Here \mathcal{W} is the subspace of \mathcal{V} which we found in Theorem 1.1. Since $\mathcal{W} \cap \mathcal{K} = \emptyset$, we know that $\mathcal{W} \cap \mathcal{K}_0 = \emptyset$. Then we obtain:

$$\#\mathcal{B}_k^1 + \#\mathcal{B}_k^2 \geq \dim(\mathcal{W} + \mathcal{K}_0) = \dim(\mathcal{W} \oplus \mathcal{K}_0) = \dim \mathcal{W} + \dim \mathcal{K}_0.$$

Since $\#\mathcal{B}_k^1 = \dim \mathcal{K}_0$, we finally obtain:

$$\#\mathcal{B}_k^2 \geq \dim \mathcal{W}.$$

□

5 Loose Ends

By virtual of Corollary 1.2, it suffices to find the lower bound for the dimension of \mathcal{V} , the subspace of $\mathcal{S}_k(\Gamma_2)$ spanned by $\{\mathbb{P}_{nI_2} | n \in \mathbb{N}\}$. We can establish the following result:

Theorem 5.1. Let \mathcal{V} be the vector space spanned by \mathbb{P}_{nI_2} . Let $\epsilon > 0$. Then for any $\epsilon > 0$, we can find sufficiently large k such that

$$\dim \mathcal{V} \geq \left(\frac{1}{2\sqrt{2}e\pi} - \epsilon \right) k.$$

The proof could be quite similar to that of Theorem 1.1: for any $\epsilon > 0$, we set

$$Q = \left\{ n \in \mathbb{N} \mid n \leq \left(\frac{1}{2\sqrt{2}e\pi} - \epsilon \right) k \right\}.$$

Then we need to show that elements in the set $\{\mathbb{P}_{nI_2} \mid n \in Q\}$ are linearly independent. Let N be the largest element in Q .

Similar to Theorem 1.1, we want to investigate the matrix

$$\mathcal{A} = (A(\mathbb{P}_{mI_2}, nI_2))_{m,n \in Q}.$$

However, we need to normalize the Fourier coefficients of \mathbb{P}_{nI_2} . That means, we define another matrix

$$\tilde{\mathcal{A}} = (\tilde{a}_{m,n})_{m,n \in Q}$$

with

$$\tilde{a}_{m,n} = \frac{\det(mI_2)^{l/2}}{\det(nI_2)^{l/2}} a_{m,n} = \frac{\det(mI_2)^{l/2}}{\det(nI_2)^{l/2}} A(\mathbb{P}_{mI_2}, nI_2).$$

Notice that $\tilde{\mathcal{A}}$ is invertible if and only if \mathcal{A} is invertible since $\tilde{\mathcal{A}}$ is equal to \mathcal{A} multiplied by two (invertible) diagonal matrices on two sides.

By the idea of Lemma 2.3, we can prove:

$$\tilde{a}_{m,n} = 8\delta_{m,n} + O\left(\frac{4\sqrt{2}\pi cN}{l}\right)^l = 8I_N + o\left(\frac{1}{k^{100}}\right).$$

when k is sufficiently large. Here c is the constant in (2.1) and we can choose c quite close to $\frac{e}{2}$. The left part is similar to Theorem 1.1

To prove this, we need the following lemma, which can be considered as a refined version of Lemma 2.2 in a special case:

Lemma 5.2. Let C be an invertible 2×2 matrix. Then the smallest eigenvalue of $C^{-1t}C^{-1}$ is $\leq 2\|C\|^{-2}$, where $\|\cdot\|$ denotes the matrix norm $\|C\| = \text{Tr}(C^t C)$.

Proof This can be obtained by direct calculation. \square

However, this result is not quite interesting since Theorem 5.1 fails to exclude Saito-Kurokawa lifts. Indeed, by the construction of Saito-Kurokawa lifts, Waldspurger's theorem and the Iwaniec-Sanark result [IS], (Or one can refer to Theorem 26.1 in [IK]. The condition $k \equiv 0 \pmod{4}$ is not essential in the theorem once the global root number is 1.) we can show that

$$\#\mathcal{B}_k^1 = \#\{F \in \mathcal{B}_k \mid F \text{ is a Saito-Kurokawa lift and } A(F, I_2) \neq 0\} \geq \left(\frac{1}{12} - \epsilon \right) k.$$

This is greater than the lower bound in Theorem 5.1. So a natural question is, how to find better lower bounds for $\dim \mathcal{V}$.

Acknowledgement

The author is grateful to the referee for a thorough reading of this paper and for helpful comments and suggestions, which lead to the improvements of the paper. In particular, the author would like to thank the referee for the valuable comments for Theorem 1.1 and Corollary 1.3.

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